Enrollment No: ______________________ **Exam Seat No:**______________________

C.U.SHAH UNIVERSITY

Winter Examination-2021

Subject Name: Theories of Ring and Field

Subject Code: 5SC03TRF1 **Branch: M.Sc. (Mathematics)**

Semester: 3 **Date: 16/12/2021** **Time: 02:30 To 05:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Give example of division ring which is not field. (02)

b. What is characteristic of Z_p? Where p is prime. (01)

c. Define: Field. (01)

d. Define: Maximal ideal. (01)

e. Is Gaussian ring a field? (01)

f. Give an example of ring which is Integral domain but not field. (01)

Q-2 **Attempt all questions** **(14)**

a. If $'a'$ is an element in a commutative ring R with unity then prove that the set $S = \{ra \ |r \in R\}$ is a principal ideal of R generated by a. (06)

b. Prove that every Euclidean ring is a principal ideal ring. (05)

c. Prove that every field is a Euclidean ring. (03)

OR

Q-2 **Attempt all questions** **(14)**

a. Let R be a Euclidean ring and let $a, b \in R$ not both zero then prove that a and b have gcd $'d'$ that can be expressed as $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$. (05)

b. If any two nonzeroelements$a$ and b in a Euclidean ring R then prove that they have L.C.M. in R. (05)

c. Prove that Gaussian ring Z[i] is a commutative ring with unity under the (04) operation '+' and '.'

Q-3 **Attempt all questions** **(14)**

a. Let R be a Euclidean ring then prove that every element in R is either unit in R or can be written as the product of a finite number of prime elements of R. (05)

b. Prove that characteristic of an integral domain is either 0 or prime. (05)

c. Give an example of ring which is division ring but not a field. (04)

OR

Q-3 **Attempt all questions** **(14)**

a. State and prove Unique factorization theorem. (06)

b. Let $R = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \neq 0, a, b, c, d \in Z\right\}$ and $I = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \neq 0, a, b \in Z\right.$. Prove that I is neither left ideal nor right ideal. (04)

c. Let R be a ring and I_1, I_2 be two ideals then prove that $I_1 + I_2$ is ideal in R. (04)

SECTION – II

Q-4 **Attempt the Following questions** **(07)**

a. Show that the polynomial (02)

$f(x) = 1 - x + x^2 - x^3 + \cdots + x^{p-1}$ where $p > 1$ is irreducible over Q.

b. Define: Extension field. (01)

c. Define: Splitting field. (01)

d. State factor theorem. (01)

e. How many elements are in $Z_2[x] / < x^2 + x + 1 >$? (01)

f. Is $f(x) = x^5 + 9x^4 + 12x^2 + 6$ reducible over Q? (01)

Q-5 **Attempt all questions** **(14)**

a. State and prove Eisentein criterion. (07)

b. State and prove Gauss Lemma. (04)

C. Prove that $F[x]$ is Euclidean ring. (03)

OR

Q-5 **Attempt all questions** **(14)**

a. Let K be an extension field of F. Then an element $a \in K$ is algebraic over F if and only if $F(a)$ is a finite extension of F. (07)

b. Let K be an extension field and $a \in K$ be algebraic over F.Suppose a satisfy an irregular polynomial $p(x) \in F[x]$ then prove that $p(x)$ must be (07) a minimal polynomial for $'a'$ over F

Q-6 **Attempt all questions** **(14)**

a. Let F be a field and K be an extension field . Prove that a polynomial of degree n over a field can have atmost$n$ roots in any extension field. (07)

b. In usual notation find $G(C.R)$. (04)

c. Prove that $G(K, F)$ is a subgroup of $Auto(K)$. (03)

OR

Q-6 **Attempt all Questions** **(14)**

a. If L is algebraic extension of K and K is algebraic extension of F then prove that L is algebraic extension of$F$ (07)

b. Prove that p^{th} cyclotomic polynomial is irreducible over Q where $p$is prime. (04)

c. Prove that every finite extension K of a field F is algebraic. (03)